Some Observations on the Spectra of Positive Operators on Finite-Dimensional C^*-Algebras

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ABSTRACT

The classical theorems of O. Perron and G. Frobenius about spectral properties of matrices with positive entries have been studied and generalized by various authors. In the book of A. Berman and R. J. Plemmons [3] the finite-dimensional aspects of this theory are described, whereas in the two monographs of H. H. Schaefer [20, 21] the infinite-dimensional theory is developed. Our purpose is to continue these extensions to finite-dimensional C^*-algebras, obtaining a complete description of the spectrum of positive irreducible operators on such ordered vector spaces.

1. PRELIMINARIES

In our paper A is always a finite-dimensional C^*-algebra and $\mathbf{1}$ the unit of A. $A_+ := \{x^*x : x \in A\}$ is the positive cone of A which is generating and normal and contains $\mathbf{1}$ as interior point. For more properties of A_+ see Pedersen [16, 1.3]; it should be noted that A_+ is a lattice cone iff A is commutative, i.e. $A = \mathbb{C}^n$ for some $n \in \mathbb{N}$. If $T \in L(A)$, $L(A)$ the vector space of all linear operators on A, then T is called positive (in symbols $T \geqslant 0$) if $T(A_+) \subseteq A_+$. For such a T it is known that the spectral radius $r(T)$ is a pole of the resolvent of maximal order on the spectral circle $\{\lambda \in \sigma(T) : |\lambda| = r(T)\}$, $\sigma(T)$ the spectrum of T, and there exists a $0 \neq x \in A_+$ such that $Tx = r(T)x$ (Berman and Plemmons [3, I.3.2]).

If A is commutative, then for $r \cdot \alpha \in \sigma(T)$, $r = r(T)$, $|\alpha| = 1$, one has $r \cdot \Gamma_\alpha \subseteq \sigma(T)$, where Γ_α is the cyclic subgroup of the circle group Γ generated by α. In particular, this implies that α is a root of unity when $r \neq 0$ (see, e.g. Schaefer [21, I.2.7]). But this result cannot be extended to the noncommutative case. To see this, let A be the algebra of all $n \times n$ matrices over $\mathbb{C}$, and let $\mathbf{1} \neq u \in A$

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be unitary. If $T \in L(A)$ is the positive mapping $x \mapsto uxu^*$ ($x \in A$), then $\sigma(T) = \{\lambda \cdot \mu^* : \lambda, \mu \in \sigma(u)\}$ and may be noncyclic. Therefore we restrict our considerations to the class of irreducible operators, where $0 \leq T \in L(A)$ is called *irreducible* if no face of A_+, distinct from $\{0\}$ and A_+, is invariant under T. Recall that a *face* F is a subcone of A_+ satisfying the following condition: $0 \leq y \leq x$, $x \in F$, $y \in A$ implies $y \in F$. Note that every face in a finite-dimensional C^*-algebra is closed and there exists a projection $p \in A$ such that $F = pA_+ p$, where $p \in A$ is called a *projection* if $p = p^*$ and $p^2 = p$ (Pedersen [16, 1.5]). For $x \in A_+$ let $F_x := \bigcup_{n=1}^{\infty} n[0, x]$, where $[0, x] := \{z \in A : 0 \leq z \leq x\}$. Then F_x is a face in A_+, and we call F_x the face generated by x.

To carry out our analysis of positive irreducible operators on finite-dimensional C^*-algebras we need some facts from the theory of Jordan algebras; we refer primarily to the book of Braun and Koecher [4] and to the recent article of Alfsen, Shultz, and Størmer [1]. By definition a *formally real* finite-dimensional Jordan algebra $(J, \circ)$ if a finite-dimensional, commutative, but not necessarily associative algebra over the reals with identity, such that $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ $(x, y \in J)$, where $x^2 = x \circ x$, and if $\sum_{i=1}^{n} x_i^2 = 0$ then $x_i = 0$ $(i = 1, \ldots, n)$. J is called a *JB-algebra* if there exists a norm on J satisfying the following requirements for $x, y \in J$:

(a) $\|x \circ y\| \leq \|x\| \cdot \|y\|$,
(b) $\|x^2\| = \|x\|^2$,
(c) $\|x^2\| \leq \|x^2\| + \|y^2\|$.

Thus A^{sa}, the self-adjoint part of A, is endowed with the product $x \circ y := \frac{1}{2}(xy + yx)$, $x, y \in A^{sa}$, a *JB*-algebra.

For the following we have to introduce a special class of *JB*-algebras, the so-called *abstract spin factors.* For this we consider a real Hilbert space H with scalar product $(\ |\ )$, norm $\|\ \|_2$, and choose an arbitrary vector $v \in H$ with $\|v\|_2 = 1$. Then $H = \{v\}^{\perp} \oplus \mathrm{lin}\{v\}$, where $\mathrm{lin}\{v\}$ is the linear subspace of H generated by $\{v\}$. Next we define on H a product $\circ$ as follows:

$$(\alpha v + \xi) \circ (\beta v + \eta) := (\alpha\beta + (\xi \mid \eta))v + (\alpha\eta + \beta\xi),$$

where $\alpha, \beta \in \mathbb{R}$, $\xi, \eta \in \{v\}^{\perp}$. Then $(H, \circ)$ is a Jordan algebra and a *JB*-algebra with respect to the norm $\|\alpha v + \xi\| := |\alpha| + \|\xi\|_2$. The positive cone of this *JB*-algebra is given by the set of $\alpha v + \xi \in H$ with $\|\xi\|_2 \leq \alpha$ (also sometimes called the "ice-cream cone"), and the Jordan automorphisms are the orthogonal operators in $L(H)$ with v as fixed vector (Topping [24]).

Example 1.1. The identity map I on A is irreducible iff $A = \mathbb{C}$. Henceforward we assume $\dim A \geq 2$.

EXAMPLE 1.2. Let M_n be the C^*-algebra of all complex $n \times n$ matrices, and let $1 \neq u \in M_n$ be unitary. Then the *-automorphism $T: x \mapsto uxu^*$ is not irreducible. To see this choose a projection $p \in M_n$ with $pu = \lambda p$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then $Tp = p$; hence the face F_p is T-invariant and nontrivial for $1 \neq p$.

EXAMPLE 1.3. Let $A = M_2$. Then A^{sa} is a JB-algebra, and with the aid of the isomorphism

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} + i\alpha_{21} \\ \alpha_{12} - i\alpha_{21} & \alpha_{22} \end{pmatrix} \mapsto \tfrac{1}{2}(\alpha_{11} + \alpha_{22}) + \left(\tfrac{1}{2}(\alpha_{11} - \alpha_{22}), \alpha_{12}, \alpha_{21}\right), \quad (*)$$

where $\alpha_{i,j} \in \mathbb{R}$, we see that A^{sa} is isomorphic to the spin factor $\operatorname{lin}\{1\} \oplus \mathbb{R}^3$. On $\mathbb{R}^3$ we consider an orthogonal operator U with $\sigma(U) = \{-1, \lambda, \lambda^*\}$, where $\lambda \in \mathbb{C} - \mathbb{R}$, and define $T \in L(H)$ as $T := I \oplus U$. Then T is positive, and no nontrivial face of H_+ will be invariant under T. To see this we first note that the means $T_n := (1/n)\sum_{i=0}^{n-1} T^i$ converge in $L(H)$ to a projection P with $P(H) = \operatorname{Fix}(T) := \{\xi \in H : T\xi = \xi\}$, since T is orthogonal (see, e.g., Halmos [11, §92]). For the unique linear form τ on H with $\tau(1) = 1$ and $\tau(\{1\}^\perp) = 0$ we have ${}^tT\tau = \tau$, from which it follows that $P\xi = \tau(\xi)1$ for all $\xi \in H$. Now if F is a face in H_+ and if $0 < \xi \in F$, then $0 < \tau(\xi)1 = \lim_{n \to \infty} T_n(\xi) \in F$; hence $F = H_+$. Since $A = A^{sa} \oplus iA^{sa}$ and since the isomorphism (*) is also an order isomorphism, we get an operator on A which is positive, is irreducible, and has the set $\{1, -1, \lambda, \lambda^*\}$ as its spectrum. This example shows that in general the peripheral spectrum of an irreducible positive operator is not a subgroup of the circle group, in contrast to the commutative finite-dimensional situation.

EXAMPLE 1.4. Again we consider the C^*-algebra M_2 and let $T \in L(A)$ be the mapping $x \mapsto \operatorname{Tr}(x)1 - x$, where Tr is the trace on M_2. Then the characteristic equation of T is $\Delta(\lambda) = (\lambda + 1)^3(\lambda - 1)$, and the minimal polynomial $m(\lambda) = (\lambda + 1)(\lambda - 1)$. A similar argument to the above shows the irreducibility of T, but in contrast to the commutative (finite-dimensional) situation, -1 is not a simple root of the characteristic equation (though it is a simple pole).

Since our C^*-algebra is finite-dimensional, there exists a normalized, faithful trace functional Tr on A, i.e. $\operatorname{Tr}(1) = 1$, $\operatorname{Tr}(z) = 0$ for $z \in A_+$ implies $z = 0$ and $\operatorname{Tr}(xy) = \operatorname{Tr}(yx)$ for every $x, y \in A$. Thus we can identify A^*, the dual space of A, with A under the pairing

$$\langle x, y \rangle := \operatorname{Tr}(xy) \qquad (x, y \in A), \tag{$\star$}$$

and we let ${}^tT$ denote the adjoint of $T \in L(A)$ under the identification ($\star$), i.e.

$\langle x, Ty \rangle = \langle {}^tTx, y \rangle$ and ${}^t({}^tT) = T$. Then $\sigma(T) = \sigma({}^tT)$, and since for any face $F \subseteq A_+$ and its polar face $F^0 := \{z \in A_+ : \langle x, z \rangle = 0$ for all $x \in F\}$ the relations $T(F) \subseteq F$ and ${}^tT(F^0) \subseteq F^0$ are equivalent, T is irreducible iff ${}^tT$ has this property. In this case $r(T)$ is a simple eigenvalue for both T and ${}^tT$, and each of these operators possesses a strictly positive eigenvector pertaining to $r(T)$ (see, e.g., Berman and Plemmons [3, I.3.23]).

2. THE MAIN THEOREM

We are now prepared for the announced extension of the Perron-Frobenius theorem to the noncommutative setting. Recall that the *peripheral spectrum* of T is the set of all eigenvalues λ satisfying $|\lambda| = r(T)$ and that a *JB*-algebra with only trivial ideals is called *simple*.

PROPOSITION 2.1. *Let A be a finite-dimensional C^*-algebra, let $0 \leqslant T \in L(A)$ be irreducible with $T1 = 1$, and let M be the linear span of all eigenvectors of T pertaining to the peripheral eigenvalues.*

(a) *$M^{sa} := M \cap A^{sa}$ is a JB-subalgebra of A^{sa}, and $T|_{M^{sa}}$ is a Jordan automorphism on M^{sa}.*

(b) *There exists a $k \geqslant 1$ and (up to a permutation) uniquely determined simple Jordan ideals $J_0, J_1, \ldots, J_{k-1}$ in M^{sa} such that $M^{sa} = \oplus_{i=0}^{k-1} J_i$, $T_0(J_i) = J_{i+1}$ $(0 \leqslant i \leqslant k-1$, $J_k = J_0)$, and J_0 is isomorphic either to $\mathbb{R}$ or to the spin factor $\mathbb{R} \oplus \mathbb{R}^s$ for some $s \geqslant 2$.*

Proof. Given $0 \leqslant T \in L(A)$, one has $\|T\| T(a^2) \geqslant T(a)^2$ for all $a \in A^{sa}$ by a famous result of Kadison [13]. It follows that

$$\|T\| T(x^*x + xx^*) \geqslant T(x)^*T(x) + T(x)T(x)^* \qquad (x \in A), \tag{1}$$

since $(x + x^*)$ and $i(x - x^*)$ are self-adjoint. If one defines, for $x, y \in A$, $x \circ y := \frac{1}{2}(xy + yx)$, then (1) reads

$$\|T\| T(x^* \circ x) \geqslant T(x)^* \circ T(x) \qquad (x \in A). \tag{2}$$

Since $\|T\| = \|T1\| = 1$, the mapping

$$B := ((x, y) \mapsto T(x^* \circ y) - T(x)^* \circ T(y)) : A \times A \to A \tag{3}$$

is positive, is sesquilinear, and satisfies $B(x, x) = 0$ for some $x \in A$ iff $B(x, y) = 0$

for every $y \in A$. To see this one has only to note that for $z \in A_+$ the mapping $(x, y) \mapsto \langle B(x, y), z \rangle$ is a positive semidefinite scalar product on A. Hence $\langle B(x, x), z \rangle = 0$ iff $\langle B(x, y), z \rangle = 0$ for every $y \in A$ by the Cauchy-Schwarz inequality. This implies the assertion, since A_+ is generating.

Let λ be a peripheral eigenvalue with eigenvector x_λ. Then $T(x_\lambda^* \circ x_\lambda) \geqslant T(x_\lambda^*) \circ T(x_\lambda) = x_\lambda^* \circ x_\lambda$. Since 1 is the strictly positive fixed vector of ${}^tT$, we have $T(x_\lambda^* \circ x_\lambda) = x_\lambda^* \circ x_\lambda = T(x_\lambda^*) \circ T(x_\lambda)$ because of $0 \leqslant \langle x_\lambda^* \circ x_\lambda - T(x_\lambda^* \circ x_\lambda), 1 \rangle = 0$. Consequently, $B(x_\lambda, x_\lambda) = 0$, and therefore $B(x_\lambda, y) = 0$ for every $y \in A$. Thus if α, β are peripheral eigenvalues of T with eigenvectors x_α, x_β, we obtain $T(x_\alpha^* \circ x_\beta) = T(x_\alpha^*) \circ T(x_\beta)$. So if M is the linear span of all the eigenvectors of T pertaining to the peripheral eigenvalues, M is closed under the product $\circ$, and hence $M^{sa} := M \cap A^{sa}$ is a finite-dimensional JB-algebra. Furthermore, $T_0 := T|_{M^{sa}}$ is a Jordan automorphism.

Since M^{sa} is finite-dimensional, there exists a maximal $k \geqslant 1$, uniquely determined (up to a permutation) simple Jordan ideals $J_0, J_1, \ldots, J_{k-1}$, and projections $p_0, p_1, \ldots, p_{k-1}$ in M^{sa}, such that $J_i = M^{sa} \circ p_i$ $(i = 0, 1, \ldots, k-1)$ and $M^{sa} = \bigoplus_{i=0}^{k-1} J_i$ (Braun and Koecher [4, I.8.3]). Since T_0 is a Jordan automorphism, the Jordan ideals $T_0(J_i)$ are also simple, hence equal to some J_j, $j \in \{i : 0 \leqslant i \leqslant k-1\}$. Therefore we can assume, by the irreducibility of T_0, that $T_0(J_0) = J_1$, $T_0(J_1) = J_2, \ldots, T_0(J_l) = J_0$ for some $0 \leqslant l \leqslant k-1$. Suppose we have $l < k-1$; then for $p := \sum_{i=0}^{l} p_i$ we have $1 \neq p$, $p > 0$, and $T_0(p) = p$. Since p is not strictly positive, F_p is a nontrivial T-invariant face. To avoid this contradiction we must have $l = k-1$.

By the results of Jordan, von Neumann, and Wigner [12, Fundamental Theorem 2], noting that the product $\circ$ is defined with the aid of an associative product, we have that J_0 is isomorphic to $\mathbb{R}$, or to $M_n^{sa}(\mathbb{K})$ $(n \geqslant 3)$, where $\mathbb{K}$ is the field $\mathbb{R}, \mathbb{C}$ or the Hamilton quaterion division algebra $\mathbb{H}$, or to the spin factor $H = \mathbb{R} \oplus \mathbb{R}^s$, $s \geqslant 2$.

Case 1: $J_0 = \mathbb{R}$. Then $J_i = \mathbb{R}$ for all $0 \leqslant i \leqslant k-1$, and T_0 is given by an irreducible permutation matrix. In this case the peripheral spectrum of T is of the form Γ_k, where Γ_k is the group of all kth roots of unity.

Case 2: $J_0 = M_n^{sa}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $n \geqslant 3$. Since $S := T_0^k$ is a Jordan automorphism on J_0, there exists a bijective, linear mapping U on the vector space $\mathbb{K}^n$ such that for all $x \in J_0$ we have $Sx = UxU^{-1}$ or $Sx = U({}^tx)U^{-1}$, where ${}^tx$ is the transpose of the matrix x (Dieudonné [7]). In the first case we choose a projection $p \in J_0$, $0 \neq p \neq p_0$, with $UpU^{-1} = p$. Then for all $0 \leqslant i \leqslant k-1$, $T_0^i(p)$ is a projection $\neq p_i$ in J_i; hence $q := \sum_{i=0}^{k-1} T_0^i(p)$ is positive and $\neq 1$ in A with $Tq = q$. Since the nontrivial face F_q is T-invariant, we obtain a contradiction. In the second case we get a contradiction in the same way. Hence $J_0 = M_n^{sa}(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is impossible.

Case 3: $J_0 = \operatorname{lin}\{p_0\} \otimes \mathbb{R}^s$ for some $s \geqslant 2$ and the proposition is proved. ■

Motivated by the commutative situation, we call the number k appearing in the second part of Proposition 2.1 the *index of imprimitivity* of T. Note that k is not the number of points in the peripheral spectrum of T (compare Example 1.3, where $k=1$).

LEMMA 2.2. *Let $0 \leq T \in L(A)$ be irreducible. Then $r = r(T) > 0$, and $r^{-1}T$ is similar to an irreducible $0 \leq S \in L(A)$ with $S\mathbf{1} = \mathbf{1}$.*

Proof. If $r=0$, then $0 = \langle Tx, y \rangle$ for all $x \in A_+$ and some interior point $y \in A_+$; hence $T=0$. By the irreducibility of T there exists a strictly positive, thus invertible, element $x \in A_+$ with $r^{-1}Tx = x$. Then the operator

$$Sz := r^{-1}x^{-1/2}T(x^{1/2}zx^{1/2})x^{-1/2}$$

satisfies $S\mathbf{1} = \mathbf{1}$, and S is similar to $r^{-1}T$ via $S = r^{-1}U^{-1}TU$, where U is the positive mapping $z \mapsto x^{1/2}zx^{1/2}$. The rest of the assertion is obvious. ∎

THEOREM 2.3. *Let A be a finite-dimensional C^*-algebra, and let $0 \leq T \in L(A)$ be irreducible with index of imprimitivity k and spectral radius r.*

(a) *There exist peripheral eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\lambda_i \notin r\Gamma_k$ $(1 \leq i \leq m)$, Γ_k the group of all kth roots of unity, such that*

$$\sigma(T) \cap r\Gamma = \left(\bigcup_{i=1}^{m} \lambda_i \Gamma_k \right) \cup r\Gamma_k.$$

(b) *Each $\lambda \in r\Gamma_k$ is a simple root of the characteristic equation (hence a simple eigenvalue) of T.*

(c) *The spectrum of T is invariant under the group of rotations (of the complex plane) corresponding to Γ_k.*

Proof. By Lemma 2.2 we can assume $T\mathbf{1} = \mathbf{1}$. Let M be the linear span of all eigenvectors pertaining to the peripheral eigenvalues of T. Then by Proposition 2.1 there exist projections $p_0, p_1, \ldots, p_{k-1}$ in the Jordan algebra M^{sa}, such that $Tp_i = p_{i+1}$ $[i \in \mathbb{Z}/(k)]$ and $p_i \circ p_j = \delta_{ij} p_i$, where δ_{ij} is the Kronecker symbol. Since $B(p_i, p_i) = 0$, we have $B(p_i, x) = 0$ for every $x \in A$ by Equation (3) in the proof of Proposition 2.1. Thus $T(x \circ p_i) = T(x) \circ p_{i+1}$ for every $x \in A$.

Given a primitive kth root of unity $\varepsilon \in \Gamma_k$. For $x \in A$ we let $x_i := x \circ p_i$ $(0 \leq i \leq k-1)$ and define $Ux := \sum_{i=0}^{k-1} \varepsilon^{-(i+1)} x_i$. Because $\sum_{i=0}^{k-1} p_i = \mathbf{1}$, U is linear and bijective with inverse $U^{-1}(x) = \sum_{i=0}^{k-1} \varepsilon^{(i+1)} x_i$. Then an easy computation shows, using $T(x \circ p_i) = T(x) \circ p_{i+1}$, that $\varepsilon T(x) = (U^{-1}TU)(x)$ for every $x \in A$. Therefore T is similar to εT; hence the assertions are fulfilled, since 1 is a simple root of the characteristic equation of T and $\sigma(T) = \sigma(U^{-1}TU) = \sigma(\varepsilon T) = \varepsilon \cdot \sigma(T)$. ■

Proposition 2.1 shows that the asymmetrical behavior of the peripheral spectrum is caused by the spin factors, whereas for positive operators the spectrum is always symmetric with respect to the real axis. For a study of positive operators on the finite-dimensional spin factors we refer to the article of Loewy and Schneider [15].

To avoid this asymmetry, we study a special class of positive operators. Recall that an operator $T \in L(A)$ is called *2-positive* if $T \otimes I_2$ is positive on the C^*-algebra $A \otimes M_2$, I_2 denoting the identity on M_2. For a discussion of 2-positive operators see Choi [5, 6]. For the following proposition compare also Evans and Høegh-Krohn [8, p. 352].

THEOREM 2.4. *Let A be a finite-dimensional C^*-algebra, and let $T \in L(A)$ be 2-positive irreducible with index of imprimitivity k and spectral radius r.*

(a) *The peripheral spectrum of T is of the form $r \cdot \Gamma_k$.*

(b) *A is the direct sum of k left ideals J_i such that $T(J_i) \subseteq J_{i+1}$, where $i = 0, 1, \ldots, k-1$ and $J_k = J_0$.*

Proof. (a): As in Lemma 2.1, we define for a strictly positive fixed vector $z \in A$ of T the operator $S \in L(A)$ by $Sx = r^{-1}z^{-1/2}T(z^{1/2}xz^{1/2})z^{-1/2}$ $(x \in A)$ [i.e., $S = r^{-1}U^{-1}TU$, where U is the (positive) operator $x \mapsto z^{1/2}xz^{1/2}$], and prove the assertion for S. By the positivity of $T \otimes I_2$, $(U \otimes I_2)^{-1}$, and $U \otimes I_2$ on $A \otimes M_2$, we see that $S \otimes I_2$ is positive too. Thus S is 2-positive and satisfies the inequality $S(x^*x) \geq S(x)^*S(x)$ for all $x \in A$ (Choi [6]). Considering the mapping $B(x, y) = S(x^*y) - S(x)^*S(y)$ from $A \times A$ into A, we have $B(x, x) = 0$ for some $x \in A$ iff $B(x, y) = 0$ for every $y \in A$. If x_α, x_β are eigenvectors of S belonging to the peripheral eigenvalues α, β then in the same manner as in the proof of Proposition 2.1 [Equation (3)] we see that $S(x_\alpha^* x_\beta) = S(x_\alpha)^*S(x_\beta) = \alpha^*\beta x_\alpha^* x_\beta$. Thus assuming $\|x_\alpha\| = 1$, we get x_α unitary, since $x_\alpha^* x_\alpha$ is a fixed vector of S, and hence must be equal to $\mathbf{1}$ by the normalization condition. Therefore $x_\alpha^* x_\beta \neq 0$, and the peripheral spectrum of S is a subgroup of the circle group, hence equal to the group of all kth roots of unity.

(b): First we assume $T\mathbf{1} = \mathbf{1}$. By Proposition 2.1 there exist projections $p_0, p_1, \ldots, p_{k-1}$ in the Jordan algebra M^{sa} such that $T(p_i) = p_{i+1}$ $(i \in \mathbb{Z}/(k))$. Since the p_i's are mutually orthogonal projections in A with $\sum_{i=0}^{k-1} p_i = \mathbf{1}$, A is the direct sum of the left ideals $J_i := Ap_i$. Since $B(p_i, p_i) = 0$, $B(\ ,\ )$ the bilinear mapping in the last paragraph, we have $T(xp_i) = T(x)T(p_i) =$

$T(x)p_{i+1}$; thus $T(J_i) \subseteq J_{i+1}$ for $i \in \mathbb{Z}/(k)$. For the general case note that $T = rUSU^{-1}$, where $S1 = 1$ and U is an algebra automorphism on A. ∎

EXAMPLE 2.1. Since on a commutative C^*-algebra every positive operator is 2-positive (Choi [5]), the results above include the classical theorems of Perron and Frobenius (see, e.g. Schaefer [21, I.6.5])

EXAMPLE 2.2. For the operator in Example 1.3, we have $k=1$. This example shows that also in this case the peripheral spectrum may be a subgroup of the circle group.

EXAMPLE 2.3. Example 1.4 shows that the conclusion of Theorem 2.3(b) may be wrong for $\lambda \notin r \cdot \Gamma_k$.

EXAMPLE 2.4. Let K be the set $\{1,2,\ldots,n\}$, n a natural number, let π be a cyclic permutation of length n on K, and let $S \in L(M_2)$ be as in Example 1.3. We consider the C^*-algebra $A = C(K, M_2)$ of all M_2-valued functions on K equipped with pointwise addition and multiplication, and the operator $T \in L(A)$ given by

$$T(f)(k) = S(f(\pi(k))) \qquad (f \in A, \quad k \in K).$$

Then T is irreducible on A. To see this, we first note that A is isomorphic (as a C^*-algebra) to $\mathbb{C}^n \otimes M_2$ and $T = P_\pi \otimes S$, where P_π is the permutation matrix corresponding to π. Then $\sigma(T) = \{\alpha\varepsilon^k : \alpha \in \sigma(S),\ k \in K\}$, where ε is a primitive nth root of unity. The eigenvectors of T are given by $\xi_i \otimes x_j$, $i \in K$, $j = 1,\ldots,4$, where ξ_i is an eigenvector belonging to P_π and x_j is an eigenvector pertaining to S. Thus $1 \otimes 1$ is the (unique) strictly positive fixed vector pertaining to T and $'T$, so T is irreducible and $\sigma(T) \cap \Gamma = \bigcup \{\alpha\Gamma_n : \alpha \in \sigma(S)\}$.

EXAMPLE 2.5. Let $S \in L(M_3)$ be given by $Sx = \frac{1}{5}[2\operatorname{Tr}(x)1 - x]$ $(x \in M_3)$. Then S is 2-positive (Choi [4, p. 522]) and $S1 = 1$. An easy computation shows that $\Delta(\lambda) = (\lambda - 1)(\lambda + \frac{1}{5})^8$ is the characteristic polynomial and $m(\lambda) = (\lambda - 1)(\lambda + \frac{1}{5})$ is the minimal polynomial of S. Since 1 is the unique fixed vector for S and $'S$, we can in the same manner as above construct an irreducible, 2-positive operator on $A = \mathbb{C}^n \otimes M_2$, which has the group Γ_n as peripheral spectrum.

REMARK 2.1. If T is 2-positive and irreducible with $T1 = 1$, then Theorem 2.4 shows that the normalized eigenvectors of T belonging to the peripheral eigenvalues form a group G. The mapping $G \to \Gamma_k$ which maps each $u \in G$

onto the corresponding eigenvalue is a homomorphism (character) with kernel $\{\alpha 1 : \alpha \in \Gamma\}$. Thus $\Gamma_k = G/\Gamma$.

REMARK 2.2. Let $T \in L(A)$ be 2-positive with $r(T) = 1$. If the fixed space of T is one-dimensional and T or ${}^tT$ has a strictly positive fixed vector, the peripheral spectrum of T is a subgroup of the circle group and its elements are simple roots of the minimal polynomial. This shows that there exist weaker conditions than irreducibility implying that the peripheral spectrum is a group.

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